

Random Serial Dictatorship versus Probabilistic Serial Rule: A Tale of Two Random Mechanisms

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Abstract

For assignment problems where agents, specifying ordinal preferences, are allocated indivisible objects, two widely studied randomized mechanisms are the Random Serial Dictatorship (RSD) and Probabilistic Serial Rule (PS). These two mechanisms both have desirable economic and computational properties, but the outcomes they induce can be incomparable in many instances, thus creating challenges in deciding which mechanism to adopt in practice. In this paper we first look at the space of lexicographic preferences and show that, as opposed to the general preference domain, RSD satisfies envyfreeness. Moreover, we show that although under lexicographic preferences PS is strategyproof when the number of objects is less than or equal agents, it is strictly manipulable when there are more objects than agents. In the space of general preferences, we provide empirical results on the (in)comparability of RSD and PS, analyze economic properties, and provide further insights on the applicability of each mechanism in different application domains.

1 Introduction

The problem of assigning a number of indivisible objects to a set of agents, each with a private preference ordering over the objects, in the absence of monetary transfer, is fundamental in many multiagent resource allocation applications, and has been the center of attention amongst researchers at the interface of artificial intelligence, economics, and mechanism design. Assigning dormitory rooms or offices to students, students to public schools, college courses to students, teaching load to faculty, organs and medical resources to patients, members to subcommittees, etc. are some of the myriad examples of one-sided matching problems [25, 28, 29, 3, 9, 22].

Two important (randomized) matching mechanisms are *Random Serial Dictatorship* (RSD) [2] and *Probabilistic Serial Rule* (PS) [6]. Both mechanisms

have important economic properties and are practical to implement. The RSD mechanism has strong truthful incentives but guarantees neither efficiency nor envyfreeness. PS satisfies efficiency and envyfreeness, however, it is susceptible to manipulation.

Therefore, since both RSD and PS induce random assignments, there are subtle points to be considered when deciding which mechanism to use. For example, given a particular preference profile, the mechanisms often produce random assignments which are simply incomparable and thus, without additional knowledge of the underlying utility models of the agents, it is difficult to determine which is the “better” outcome. Furthermore, properties like non-manipulability and envyfreeness can depend on whether there is underlying structure in the preferences, and even in general preference models it is valuable to understand under what conditions a mechanism is likely to be non-manipulable or envyfree as this can guide designers choices.

In this paper, we consider the random assignment of m objects to n agents, where each agent reports her complete preferences as a strict ordering over objects. We first look at the space of lexicographic preferences and define two axioms for fairness based on partial preference orderings. We show that, as opposed to the general preference domain, RSD satisfies envyfreeness. Moreover, we show that although under lexicographic preferences PS is strategyproof when the number of objects is less than or equal agents, it is susceptible to manipulation when there are more objects than agents.

We empirically study the space of all possible preference profiles for various matching problems, and provide insights on the comparability of RSD and PS under the various agent-object combinations with the aim of providing practical insights on the different properties of RSD and PS.

2 Relation to the literature

In the house allocation problem with ordinal preferences, Svensson showed that serial dictatorship is the only deterministic mechanism that is strategyproof, nonbossy, and neutral [30]. Abdulkadiroglu and Sonmez showed that uniform randomization over all serial dictatorship assignments, a.k.a RSD, is equivalent to the top trading cycle’s core from random initial endowment, and thus satisfies strategyproofness, proportionality, and ex post efficiency [2]. Although RSD is strategyproof and proportionally fair, but it does not guarantee envyfreeness.

Bogomolnaia and Moulin noted the inefficiency of the RSD mechanism from the ex ante perspective, and characterized random assignment mechanisms based on first order stochastic dominance (sd) [6]. They further proposed the probabilistic serial mechanism as an efficient and envyfree mechanism. While PS is not strategyproof, it satisfies weak strategyproofness for problems with equal number of agents and objects. Kojima studied random assignment of multiple indivisible objects and showed that PS is strictly manipulable (not weakly strategyproof) even when there are only two agents [18]. Kojima and Manea, later, considered a setting with multiple copies of each object and showed that in large

assignment problems with sufficiently many copies of each object, truth-telling is a weakly dominant strategy in PS [19]. In order to give a rationale for practical employment of RSD and PS in real-life applications, Che and Kojima further analyzed the economic properties of PS and RSD in large markets with multiple copies of each object and concluded that these mechanisms become equivalent when the market size becomes large [10]. Particularly, they showed that inefficiency of RSD and manipulability of PS vanishes when the number of copies of each object approaches infinity.

The practical implications of deploying RSD and PS have been the center of attention in many one-sided matching problems [1]. In the school choice setting with multi-capacity alternatives, Pathak compared RSD and PS using data from the assignment of public schools in New York City and observed that many students obtained a more desirable random assignment through PS [24]. However, the efficiency difference was quite small. These observations further confirm Che and Kojima’s equivalence result when there are multiple copies of each object (*i.e.* school seats) available [10]. Despite these findings for arbitrary large markets, the equivalence results of Che and Kojima, and its extension to all random mechanisms by Liu and Pycia [20], do not hold when the quantities of each object is limited to one [10].

This paper sets out to study the comparability of PS and RSD when there is only one copy of each object, and analyze the space of all preference profiles for different numbers of agents and objects. We make several intriguing observations about the manipulability of PS and fairness properties of RSD. Following Manea’s work on asymptotic inefficiency of RSD [21], we show that despite this inefficiency result, the fraction of random assignments at which PS stochastically dominates RSD vanishes when the number of agents is less than or equal to the available objects. Moreover, we show that this result strongly holds for lexicographic preferences when there is equal number of agents and objects.

3 Formal Representation

A one-sided matching problem $\langle N, M, \succ \rangle$ consists of a set of agents N , where $|N| = n$, a set of distinct indivisible objects M with $|M| = m$, and a *preference profile* \succ denoting the set of strict and complete preference orderings of agents over the objects. Let \mathcal{P} denote the set of all complete and strict preferences over M . Each agent i has a private preference ordering denoted by $\succ_i \in \mathcal{P}$, where $a \succ_i b$ indicates that agent i prefers object a over b . Thus, a *preference profile* is $\succ = (\succ_1, \dots, \succ_n) \in \mathcal{P}^n$. We represent the preference ordering of agent i by the ordered list of objects $\succ_i = a \succ_i b \succ_i c$ or $\succ_i = (abc)$, for short.

A random assignment is a stochastic $n \times m$ matrix A that specifies the probability of assigning each object j to each agent i . The probability vector

$A_i = (A_{i,1}, \dots, A_{i,m})$ denotes the random allocation of agent i , that is,

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,m} \\ A_{2,1} & A_{2,2} & \dots & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \dots & A_{n,m} \end{pmatrix}$$

Let \mathcal{A} refer to the set of possible assignments. An assignment $A \in \mathcal{A}$ is said to be *feasible* if and only if $\forall i \in N, \sum_{j \in M} A_{i,j} = 1$, that is, the probability distribution function is valid for each object.

Given random assignment A_i , the probability that agent i is assigned an object that is at least as good as object ℓ is defined as follows

$$w(\succ_i, \ell, A_i) = \sum_{j \in M: j \succeq_i \ell} A_{i,j} \quad (1)$$

A deterministic assignment is simply a binary matrix of degenerate lotteries over objects that allocates each object to exactly one agent with certainty. Every random assignments is a convex combination of deterministic assignments and is induced by a lottery over deterministic assignments [31].

A *matching mechanism* is a mapping $\mathcal{M} : \mathcal{P}^n \rightarrow \mathcal{A}$ from the set of possible preference profiles to the set of random assignments.

3.1 Properties

In this section we define key properties that matching mechanisms should have. In particular, we formally define efficiency, strategyproofness and envyfreeness for (randomized) matching mechanisms.

In the context of deterministic assignments, an assignment A_i *Pareto dominates* another assignment B_i at \succ if $\exists i \in N$ such that $A_i \succ_i B_i$ and $\forall k \in N$ $A_k \succeq_k B_k$, where $A_i \succ_i B_i$ denotes that agent i strictly prefers assignment A_i over B_i . An assignment is *Pareto efficient* at \succ if no other assignment exists that Pareto dominates it at \succ . Extending the Pareto efficiency requirement to random mechanisms, we focus only on mechanisms that always guarantee a Pareto efficient solution ex post. A random assignment is called *ex post efficient* if it can be represented as a probability distribution over deterministic Pareto efficient assignments.

To evaluate the quality of a random assignment, we follow the convention proposed by Bogomolnaia and Moulin based on first-order stochastic dominance [6].

Definition 1. Given a preference ordering \succ_i , random assignment A_i **stochastically dominates** (*sd*) assignment $B_i (\neq A_i)$ if

$$\forall \ell \in M, w(\succ_i, \ell, A_i) \geq w(\succ_i, \ell, B_i) \quad (2)$$

Stochastic dominance is a strong requirement. It implies that for the entire space of utility functions that is consistent with ordinal preferences, an agent's expected utility under A is always greater than her expected utility under B .

Definition 2. A random assignment is **sd-efficient** if for all agents, it is not stochastically dominated by any other random assignment.

A matching mechanism is *sd-efficient* if at all preference profiles $\succ \in \mathcal{P}^n$, for all agents $i \in N$, the induced random assignment is not stochastically dominated by any other assignment. Intuitively, no other mechanism exists that *all* agents strictly prefer its outcome to their current random assignments.

A matching mechanism is *sd-strategyproof* if there exists no non-truthful preference ordering $\succ'_i \neq \succ_i$ that improves agent i 's random assignment. More formally,

Definition 3. Mechanism \mathcal{M} is **sd-strategyproof** if at all preference profiles $\succ \in \mathcal{P}^n$, for all agents $i \in N$, and for any misreport $\succ'_i \in \mathcal{P}^n$, such that $A = \mathcal{M}(\succ)$ and $A' = \mathcal{M}(\succ'_i, \succ_{-i})$, we have:

$$\forall \ell \in M, w(\succ_i, \ell, A_i) \geq w(\succ_i, \ell, A'_i) \quad (3)$$

In the context of random mechanisms, *sd-strategyproofness* is a strict requirement. It states that under any utility model consistent with the preference orderings, no agent can improve her expected outcome by misreporting.

We can also define a milder version of strategyproofness. A mechanism is *weakly sd-strategyproof* if at all preference profiles, no agent can misreport her preferences and obtain a random assignment that strictly improves her assignment for the entire space of utility models consistent with her true preference ordering.

Definition 4. Mechanism \mathcal{M} is **weakly sd-strategyproof** if for all preference profiles $\succ \in \mathcal{P}^n$, for any agent i with misreport \succ'_i , where $A = \mathcal{M}(\succ)$ and $A' = \mathcal{M}(\succ'_i, \succ_{-i})$, we have

$$\exists \ell \in M, w(\succ_i, \ell, A_i) > w(\succ_i, \ell, A'_i) \quad (4)$$

We say that a random assignment induced by mechanism \mathcal{M} is *manipulable* if it is not *sd-strategyproof*. That is, there exists an agent i with preference \succ_i that her assignment under a truthful report does not stochastically dominate her assignment under a non-truthful report. Formally, assignment A induced by mechanism \mathcal{M} is **manipulable** at preference profile \succ if there exists an agent $i \in N$ with misreport \succ'_i such that if $A = \mathcal{M}(\succ)$ and $A' = \mathcal{M}(\succ'_i, \succ_{-i})$, we have $\exists \ell \in M, w(\succ_i, \ell, A'_i) > w(\succ_i, \ell, A_i)$.

Intuitively, an assignment is manipulable if for some utility function consistent with the ordinal preferences, an agent's expected utility under a non-truthful report improves.

An assignment induced by a matching mechanism is strictly manipulable if the mechanism does not satisfy weak *sd-strategyproofness*, i.e., the induced corresponding random assignment is *sd-manipulable*. Thus, assignment A induced by mechanism \mathcal{M} is **sd-manipulable** at preference profile \succ if there exists an agent $i \in N$ with preference \succ_i , and a misreport $\succ'_i \in \mathcal{P}^n$ such that

$A'_i = \mathcal{M}(\succ'_i, \succ_{-i})$ stochastically dominates $A_i = \mathcal{M}(\succ)$, that is, $\forall \ell \in M$, $w(\succ_i, \ell, A'_i) \geq w(\succ_i, \ell, A_i)$.

An *sd*-manipulable assignment indicates that there exists an agent that can strictly benefit from misreporting. Clearly, a weakly *sd*-strategyproof assignment is not *sd*-manipulable.

To analyze the fairness properties of matching mechanisms, we study the ordinal notion of envyfreeness. An assignment is *sd*-envyfree if each agent strictly prefers her random allocation to any other agent's assignment, that is, given \succ_i agent i 's random assignment stochastically dominates any other agent's assignment.

Definition 5. *Given agent i 's preference \succ_i , assignment A_i is **sd-envyfree** if for all agents $\forall k \neq i \in N$,*

$$\forall \ell \in M, w(\succ_i, \ell, A_i) \geq w(\succ_i, \ell, A_k) \quad (5)$$

A matching mechanism satisfies *sd*-envyfreeness if at all preference profiles $\succ \in \mathcal{P}^n$, it induces *sd*-envyfree assignments for all agents.

An assignment is **weakly sd-envyfree** if no agent strictly prefers another agent's random assignment to her own. Formally, for all agents $i, k \in N$ with $A_i \neq A_k$, we have $\exists \ell \in M$, $w(\succ_i, \ell, A_i) > w(\succ_i, \ell, A_k)$.

4 Two Random Mechanisms

In this section, we formally introduce two widely studied matching mechanisms, Random Serial Dictatorship [2] and Probabilistic Serial rule [6].

Random Serial Dictatorship (RSD) is a uniform distribution over all possible (priority) orderings of agents, and for each realization of the orderings, the first agent receives her most preferred object, the next agent receives her most preferred object among the set of remaining objects, and so on until no object remains unassigned.¹

Definition 6. *Given a preference profile, the **Random Serial Dictatorship (RSD)** mechanism is a uniform distribution over the assignments induced by the set of all possible priority orderings over agents.*

Abdulkadiroğlu and Sönmez showed the equivalence of the random serial dictatorship and the core from the random endowment in the house allocation problem, and argued that this equivalence justifies the wide use of the RSD mechanism in many practical applications such as student housing, course allocating, etc. [2].

Bogmolnaia and Moulin proposed the Probabilistic Serial rule (PS) [6]. Given a preference profile, the PS mechanism treats objects as a set of divisible

¹For $n < m$, RSD requires a careful method for picking sequence at each realized priority ordering, which will directly affect the efficiency and envy properties of the assignments [8, 17]. For simplicity, we assume that when $n < m$, at each priority ordering the first agent chooses $(m - n + 1)$ objects, and the rest of the agents choose one object each.

objects and simulates a Simultaneous Eating Algorithm (SEA): At every point in time, each agent starts eating from her top choice according to \succ_i at the unit speed. When an object is completely exhausted (eaten away), each agent eats away from her most preferred objects among the remaining objects. The eating algorithm terminates when all objects are exhausted.

The random assignment of agent i , when SEA terminates, is the fraction of each object that has been eaten away by agent i . We adopt the following definition from Bogomolnaia and Moulin [6]:

Definition 7. *Given a preference profile, the **Probabilistic Serial rule (PS)** is the random probability assignment by simulating the Simultaneous Eating Algorithm with uniform speed.*

For a given preference profile \succ , we let $PS(\succ) \in \mathcal{A}$ and $RSD(\succ) \in \mathcal{A}$ denote the outcomes of PS and RSD mechanisms respectively.

In their seminal work, Bogomolnaia and Moulin characterized the economic properties of these two mechanisms and showed that RSD does not guarantee *sd*-efficiency [6]. The following example, adopted from Bogomolnaia and Moulin (2001), illustrates the inefficiency of the RSD mechanism.

Example 1. *Suppose there are four agents $N = \{1, 2, 3, 4\}$ and four objects $M = \{a, b, c, d\}$. Consider the following preference profile $\succ = ((abcd), (abcd), (badc), (badc))$.*

	a	b	c	d
A_1	1/2	0	1/2	0
A_2	1/2	0	1/2	0
A_3	0	1/2	0	1/2
A_4	0	1/2	0	1/2

(a) Assignment under $PS(\succ)$

	a	b	c	d
A_1	5/12	1/12	5/12	1/12
A_2	5/12	1/12	5/12	1/12
A_3	1/12	5/12	1/12	5/12
A_4	1/12	5/12	1/12	5/12

(b) Assignment under $RSD(\succ)$

Table 1: Example showing the inefficiency of RSD

In this example, all agents strictly prefer the assignment induced by PS over the RSD assignment. Thus, RSD is inefficient at this preference profile.

Table 2 summarizes the theoretical results from the literature for both RSD and PS. For $n \geq m$, PS satisfies *sd*-efficiency, *sd*-envyfreeness, and weakly *sd*-strategyproofness, while RSD is *sd*-strategyproof, and weakly *sd*-envyfree, but it is not *sd*-efficient nor *sd*-envyfree [6]. However, for $n < m$, PS is *sd*-manipulable, i.e., PS is not even weakly *sd*-strategyproof [19].

4.1 The Incomparability of RSD and PS

The theoretical properties of RSD and PS do not provide proper insight into the head-to-head comparison and applicability of these two mechanisms.

RSD does not guarantee *sd*-efficiency, meaning that the assignments induced by RSD are not equivalent to the *sd*-efficient assignments induced by PS. However, in many instances of preference profiles the random assignments induced

	$n \geq m$		$n < m$	
	PS	RSD	PS	RSD
<i>sd</i> -strategyproof	weak	✓	✗	✓
<i>sd</i> -efficiency	✓	✗	✓	✗
<i>sd</i> -envyfree	✓	weak	✓	weak

Table 2: The summary of properties

	a	b	c		a	b	c
A_1	1/2	0	1/2	(a)	A_1	1/2	0
A_2	1/2	1/4	1/4		A_2	1/2	1/6
A_3	0	3/4	1/4	(b)	A_3	0	5/6

(a) Assignment under $PS(\succ)$

(b) Assignment under $RSD(\succ)$

Table 3: Incomparability of RSD and PS

by PS and RSD are simply incomparable, in that neither assignment stochastically dominates the other one. The following example illustrates this subtle distinction even for $n = m = 3$.

Example 2. Suppose there are three agents $N = \{1, 2, 3\}$ and three objects $M = \{a, b, c\}$. Consider the following preference profile $\succ = ((acb), (abc), (bac))$.

As shown in Table 3, neither mechanism provides an assignment that stochastically dominates the other: agent 1 receives the same allocation under both RSD and PS, agent 2 strictly prefers PS over RSD, and agent 3 strictly prefers RSD over PS. Thus, the two assignments are incomparable in terms of *sd*-efficiency.

In such instances, the efficiency of the assignments is ambiguous with respect to ordinal preferences. Thus, the (ex ante) efficiency of the induced assignments is contingent on the underlying von Neumann-Morgenstern (vNM) utility functions.

Similarly, the envy of RSD and the manipulability of PS depend on the structure of preference profiles, and thus, a compelling question, that justifies the practical implications of deploying a matching mechanism, is to analyze the space of preference profiles to find the likelihood of inefficient, manipulable, or envious assignments under these mechanisms.

In the next sections, we focus attention on lexicographic preference, and discuss the properties of RSD and PS in this domain. Then, we empirically study the space of all preference profiles for various matching problems.

5 Lexicographic Preferences

In many real-life scenarios, players have preferences that are lexicographic on alternatives or objects, for example political negotiations, voting problems, team

standings in sports like Hockey, sequential screening process for hiring candidates or choosing alternatives, etc. [14, 32, 13]. Lexicographic preferences are present in various applications and have been extensively studied in artificial intelligence and multiagent systems as a means of assessing allocations based on ordinal preferences [11, 26].

Given two assignments, an agent prefers the one in which there is a higher probability for getting the most-preferred object. Formally, given a preference ordering $\succ_i = o^1 \succ o^2 \dots \succ o^m$, agent i prefers any allocation A_i that assigns a higher probability to her top ranked object p_{i,o^1} over any assignment B_i with $B_{i,o^1} < A_{i,o^1}$, regardless of the assigned probabilities to all other objects. Only when two assignments allocate the same probability to the top object will the agent consider the next-ranked object.

Definition 8. *Given agent i 's preference ordering \succ_i , assignment A_i **lexicographically dominates (ld)** assignment B_i if*

$$\exists \ell \in M : A_{i,\ell} > B_{i,\ell} \wedge \forall j \succ_i \ell : A_{i,j} = B_{i,j}. \quad (6)$$

A matching mechanism is lexicographically efficient (*ld-efficient*) if for all preference profiles its induced assignment is not lexicographically dominated by any other random assignment. By definition *sd*-efficiency yields *ld*-efficiency, however, an *ld*-efficient assignment may not be *sd*-efficient. The following example illustrates this:

Example 3. *Consider four agents $N = \{1, 2, 3, 4\}$ and four objects $M = \{a, b, c, d\}$ at the following preference profile $\succ = ((cabd), (acdb), (cbda), (acbd))$. Table 4 shows the assignments induced by PS and RSD. Here, PS lexicographically dominates RSD since all the agents receive a higher probability of their more preferred objects under PS. However, PS does not stochastically dominate RSD because agent 2 (similarly agent 4) weakly prefers the RSD assignment as $w(\succ_2, c, RSD(\succ)) = \frac{11}{24} + \frac{1}{12} = \frac{13}{24}$ is greater than $w(\succ_2, c, PS(\succ)) = \frac{1}{2} + 0 = \frac{1}{2}$. Thus, the two random assignments are in fact incomparable with respect to stochastic dominance.*

Proposition 1. *PS is ld-efficient, RSD is not.*

Proof. The *ld*-efficiency of PS is directly derived from its *sd*-efficiency. Example 1 illustrates that RSD is not *ld*-efficient. \square

	a	b	c	d
A_1	0	1/3	1/2	1/6
A_2	1/2	0	0	1/2
A_3	0	1/3	1/2	1/6
A_4	1/2	1/3	0	1/6

(a) $PS(\succ)$ assignment

	a	b	c	d
A_1	1/12	1/3	5/12	1/6
A_2	11/24	0	1/12	11/24
A_3	0	5/12	5/12	1/6
A_4	11/24	1/4	1/12	5/24

(b) $RSD(\succ)$ assignment

Table 4: An example showing PS dominating RSD lexicographically but not w.r.t stochastic dominance.

Sd-efficiency implies *ld*-efficiency. However, similar to stochastic dominance relations, in terms of matching mechanisms it is unclear whether PS or RSD are comparable in terms of *ld*-efficiency, in particular at those instances of preference profiles that they are non-comparable under stochastic dominance.

5.1 Envyfreeness

In many practical situations, such as political negotiations, hiring candidates, and sports standings [14, 13], players most often associate higher regard to their top choices, and assess the fairness of random outcomes according to *ld*-envyfreeness.

Definition 9. *Given agent i 's preference \succ_i , assignment A_i is **ld-envyfree** if there exists no agent-object pair $k \in N, \ell \in M$ such that, $A_{k,\ell} > A_{i,\ell} \wedge \forall j \succ_i \ell : A_{i,j} = A_{k,j}$.*

A matching mechanism satisfies *ld*-envyfreeness if at all preference profiles $\succ \in \mathcal{P}^n$ it induces *ld*-envyfree assignments for all agents. *Sd*-envyfreeness leads to *ld*-envyfreeness; however, *ld*-envyfree allocations may not satisfy *sd*-envyfreeness.

Example 4. *In Example 2, under RSD no agent is ld-envious of any other agent. However, agent 2 is weakly envious of agent 3's assignment. Thus, the assignment is not sd-envyfree. Assume agent 2's utility for the objects is $u_2(a) > u_2(b) > u_2(c)$. Agent 2's utility for objects as good as object b under her assignment and agent 3's assignment can be written as follows $\frac{5}{6}u_2(b) > \frac{3}{6}u_2(a) + \frac{1}{6}u_2(b)$. Thus, for all utility functions at which $u_2(b) > \frac{3}{4}u_2(a)$, agent 2 prefers agent 3's random assignment.*

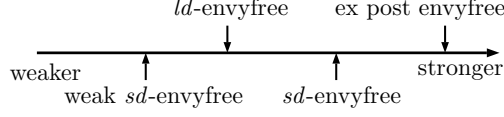
Based on the definition of stochastic dominance, it is easy to see that every *sd*-envyfree mechanism satisfies *ld*-envyfreeness. PS satisfies *sd*-envyfreeness, hence, it is *ld*-envyfree for all preference profiles. However, *ld*-envyfreeness is stronger than weak *sd*-envyfreeness.

Proposition 2. *Every ld-envyfree allocation is weakly sd-envyfree, that is, $ld\text{-envyfree} \subset weakly\ sd\text{-envyfree}$.*

The inclusion in Proposition 2 is strict: consider two agents with $\succ_1 = \succ_2 = (abc)$. The random assignments $A_1 = (\frac{1}{6}, \frac{5}{6}, 0)$ and $A_2 = (\frac{4}{6}, \frac{1}{6}, \frac{1}{6})$ are weakly *sd*-envyfree, but do not satisfy *ld*-envyfreeness. The relationship between the different definitions of fairness is as follows:²

Before analyzing the envyfreeness of RSD in the lexicographic domain, we need to define a simple axiom called *downward partial symmetry* (DPS). Intuitively, DPS is an extension to the fairness notion of equity (a.k.a proportionality). DPS states that starting from the first-ranked objects downwards, all agents with identical partial preferences receive exactly the same random assignment of the objects in the partial ordering.

²For a complete discussion on the computational complexity of finding or verifying the fairness concepts with respect to envyfreeness and proportionality see [4].



Definition 10. Let \succ_i^ℓ denote the partial preference of agent i , such that $\succ_i^\ell = o^1 \succ_i \dots \succ_i o^\ell$. A random assignment satisfies **downward partial symmetry (DPS)** if for all agent $i, k \in N$,

$$\forall \ell \text{ where } \succ_i^\ell = \succ_k^\ell \Leftrightarrow \text{for all } j \succeq_i \ell, A_{i,j} = A_{k,j} \quad (7)$$

Similarly, a random assignment satisfies **upward partial symmetry (UPS)** if starting from the *least favorite* objects, agents with identical partial preferences receive exactly the same random assignment of the objects in the partial ordering. It is easy to see that PS satisfies both UPS and DPS. However, RSD only satisfies DPS.

Proposition 3. *RSD satisfies DPS but does not guarantee UPS.*

Proof. In Example 3, agent 1 and 4's assignments illustrate the non-existence of UPS.

RSD is a uniform distribution over the set of priority orderings, and thus, satisfies equal treatment of equals for full preference orderings, i.e., $\succ_i = \succ_k$ then $A_{i,\ell} = A_{k,\ell}, \forall \ell$. Assume for contradiction that for two agents $i, k \in N$ where $\succ_i^\ell = \succ_k^\ell, \exists \ell' \succeq_i \ell$ such that $A_{i,\ell'} \neq A_{k,\ell'}$. For the sake of simplicity, let us assume $A_{i,\ell'} < A_{k,\ell'}$. For any serial dictatorship where i precedes k in the priority ordering, i chooses an object j such that $j \succ_i \ell'$. This immediately implies that $A_{i,j} > A_{k,j}$ for some $j \succ_k \ell'$. We can continue this by induction backward up to the first-ranked item, implying that there exists a priority ordering at which agent i chooses a less preferred object, that is $\succ_i^\ell \neq \succ_k^\ell$, which contradicts the assumption. Similarly we can show that if $\succ_i^\ell \neq \succ_k^\ell$, it is impossible to achieve $A_{i,j} = A_{k,j}, \forall j \succeq_i \ell$. \square

The next theorem shows that although RSD is not *sd-envyfree*, it satisfies the weaker notion of *ld-envyfree*.

Theorem 1. *RSD is ld-envyfree, for any n and m .*

Proof. Assume for contradiction that there exists an agent $k \in N$ with random assignment A_k that agent i *ld* prefers A_k to her own A_i . Therefore, by definition of lexicographic dominance, there exists an object $\ell \in M$ such that RSD assigns higher probability to k , i.e. $A_{i,\ell} < A_{k,\ell}$, and $\forall j \in M : j \succ_i \ell, A_{i,j} = A_{k,j}$. According to Proposition 3 this is only possible when $\succ_i^\ell \neq \succ_k^\ell$, implying that RSD is *ld-envyfree*. \square

The *ld-envyfree*ness of RSD is noteworthy: it shows that despite RSD is not *sd-envyfree*, it satisfies envyfreeness with respect to lexicographic preferences.

5.2 Strategyproofness

When multiple objects can be allocated to each agent, manipulating the PS outcome may result in a stochastically dominant assignment. In fact, even for $n = 2$, the random assignment induced PS is *sd*-efficient and *sd*-envyfree but not weakly *sd*-strategyproof (i.e. *sd*-manipulable) [18].

Example 5. Consider two agents and four objects with preferences $\succ = ((abcd), (bcad))$. Assume that agent 1 misreports her preference as $\succ'_1 = (bacd)$.

	a	b	c	d
A_1	1	0	1/2	1/2
A_2	0	1	1/2	1/2

(a) $PS(\succ_1, \succ_2)$ assignment

	a	b	c	d
A_1	1	1/2	0	1/2
A_2	0	1/2	1	1/2

(b) $PS(\succ'_1, \succ_2)$ assignment

Table 5: Random assignments under (a) truthful and (b) non-truthful reports.

Agent 1's allocation under the misreport stochastically dominates her assignment when reporting truthfully because for all objects ℓ , $w(\succ_1, \ell, PS(\succ'_1, \succ_2)) \geq w(\succ_1, \ell, PS(\succ_1, \succ_2))$.

The main intuition for the strong manipulability of PS comes from the eating sequence of agents. Since agents can be allocated more than one object, the sequence in which agents choose to eat away the objects becomes crucial (for a discussion on the game-theoretic properties and complexity of manipulation in picking sequences see [17, 8, 7]).

It is apparent that *sd*-manipulability leads to *ld*-manipulability, however, there are random assignments under PS that are *ld*-manipulable but not *sd*-manipulable, as demonstrated by the following inclusion relation: *sd*-manipulable \subset *ld*-manipulable \subset manipulable.

In the domain of divisible objects, PS is the only stochastic mechanism which is *sd*-efficient, *sd*-envyfree, and *ld*-strategyproof on the lexicographic preference domain [27]. However, this characterization is only valid for $n \geq m$.

Proposition 4. For $n < m$, PS is not *ld*-strategyproof nor *sd*-strategyproof.

Proof. Proof follows from Example 5. \square

Ld-strategyproofness is the direct consequence of a simple axiom called *bounded invariance property* [5], that is, no agent is able to change her preference order for any object she likes less than j such that her (probabilistic) allocation of j improves. This axiom was first characterized by Bogomolnaia and Heo, where they showed that PS satisfies bounded invariance for $n = m$ [5]. However, since for $n < m$ each agent has an eating capacity of more than 1, PS no longer satisfies the bounded invariance property.

Proposition 5. For $n < m$, PS does not satisfy the bounded invariance property.

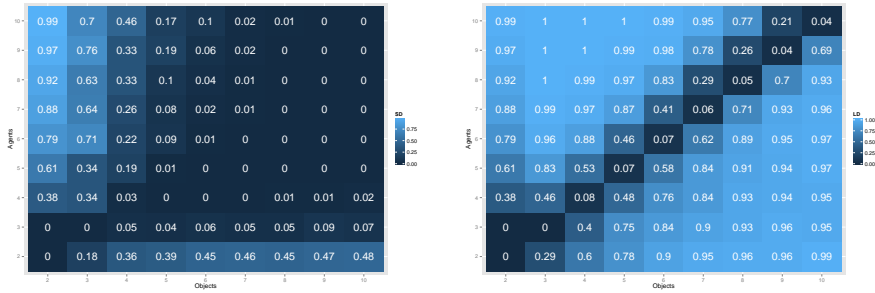
6 Experimental Results

The theoretical properties of PS and RSD, even in restricted domains such as lexicographic preferences, only provide limited insight into their practical applications. In particular, when deciding which mechanism to use in different settings, the incomparability of PS and RSD leaves us with an ambiguous choice. Thus, we examine the properties of RSD and PS in the space of all possible preference profiles.

The number of all possible preference profiles is super exponential $(m!)^n$. Thus, for each combination of n agents and m objects we considered 1,000 randomly generated instances by sampling from a uniform preference distribution. Thus, each data point is the average of 1,000 replications. For each preference profile, we ran both PS and RSD mechanisms and compared their outcomes (in terms of random assignments).

A preliminary look at our empirical results illustrates the followings: when $m \leq 2, n \leq 3$, PS coincides exactly with RSD, which results in the best of the two mechanisms, i.e., both mechanisms are *sd*-efficient, *sd*-strategyproof, and *sd*-envyfree.³ Moreover, when $m = 2$, for all n PS is *sd*-strategyproof (although the PS assignments are not necessarily equivalent to assignments induced by RSD), RSD is *sd*-envyfree, and for most instances PS stochastically dominates RSD, particularly when $n \geq 4$.

6.1 Efficiency



(a) The fraction that PS stochastically dominates RSD. (b) The fraction that PS lexicographically dominates RSD.

Figure 1: The fraction of preference profiles under which PS is superior to RSD.

Our first finding is that the fraction of preference profiles at which RSD and PS induce equivalent random assignments goes to 0 when n grows. There are two conclusions that one can draw from this observation. First, this result confirms the theoretical results of Manea on asymptotic inefficiency of RSD [21], in that,

³This was first noted by Bogomolnaia and Moulin for $n = m = 2$ [6].

in most instances, the assignments induced by RSD are not equivalent to the PS assignments. Second, this result also suggests that the incomparability of outcomes is significant, that is, the (ex ante) efficiency of the random outcomes is highly dependent on the underlying utility models. We propose the following conjecture.

Conjecture 1. *The fraction of preference profiles $\succ \in \mathcal{P}^n$ for which RSD is stochastically dominated by PS at \succ converges to zero as $\frac{n}{m} \rightarrow 1$.*

Our empirical results support Conjecture 1 on comparability of RSD and PS. Figure 1a shows that although RSD is inefficient when n grows beyond $n > 5$, due to incomparability of RSD and PS with regard to stochastic dominance relation, the RSD induced assignments are not stochastically dominated by *sd*-efficient assignments induced by PS.

We also see similar results when we restrict ourselves to lexicographic preferences (Figure 1b).

Conjecture 2. *The fraction of preference profiles $\succ \in \mathcal{P}^n$ for which RSD is lexicographically dominated by PS at \succ converges to zero as $\frac{n}{m} \rightarrow 1$.*

For lexicographic preferences, we also observe that the fraction of preference profiles for which PS assignments strictly dominate RSD-induced allocations goes to 1 when the number of agents and objects diverge.

Conjecture 3. *The fraction of preference profiles $\succ \in \mathcal{P}^n$ for which RSD is lexicographically dominated by PS at \succ converges to 1 as $|n - m|$ grows.*

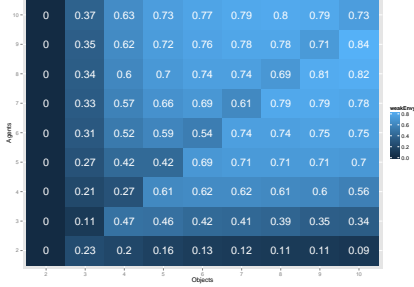
One immediate conclusion is that although RSD does not guarantee either *sd*-efficiency or *ld*-efficiency, in most settings with $n \leq m$, especially when $n = m$, neither of the two mechanisms is preferred in terms of efficiency. Hence, one cannot simply rule out the RSD mechanism.

6.2 Envy in RSD

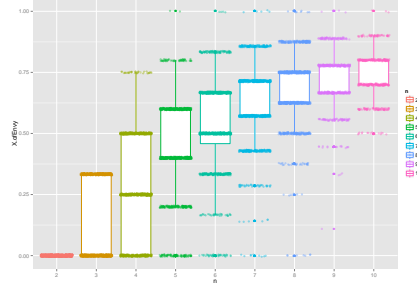
In our next experiment we measure the fraction of agents that are weakly envious of at least one another agent. Each data point represents the fraction of (weakly) envious agents.

Figure 2 shows that for RSD, the percentage of agents that are weakly envious increases with the number of agents. Figure 2a reveals an interesting observation: fixing any $n > 3$, the percentage of agents that are (weakly) envious grows with the number of objects, however, there is a sudden drop in the percentage of envious agents when there are equal number of agents and objects.

For better understanding of the population of agents who feel (weakly) envious under RSD, we illustrate the envy distribution over the set of preference profiles for each $n = m$ (Figure 2b). One observation is that there are few distinct envy profiles at each n , each representing a particular class of preference profiles, and by increasing n , the fraction of agents that are envious of at least one other agent increases.

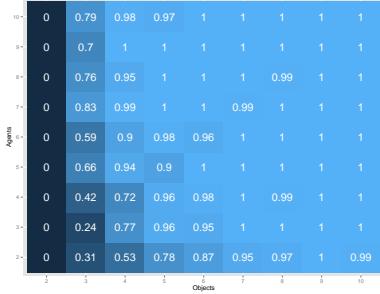


(a) A heatmap showing the percentage of envious agents.

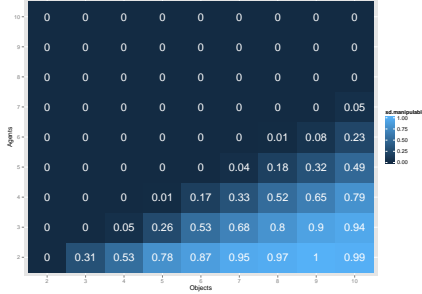


(b) Boxplots showing the various envy profiles for $n = m$. The Y axis represents the percentage of envious agents.

Figure 2: Plots representing the percentage of (weakly) envious agents under RSD.



(a) The fraction of manipulable preference profiles under PS.



(b) The fraction of sd -manipulable preference profiles under PS.

Figure 3: Heatmaps illustrating the manipulability of PS.

6.3 Manipulability of PS

In Section 5, we characterized the incentive properties of the PS mechanism with regards to the stochastic dominance and lexicographic dominance relations, arguing that although for $n \geq m$ PS is weakly sd -strategyproof and ld -strategyproof, when $n < m$ PS no longer satisfies these two properties.⁴ Nonetheless, we are interested in gaining insights on the fraction of preference profiles at which PS is manipulable.

⁴A recent experimental study on the incentive properties of PS shows that human subjects are less likely to manipulate the mechanism when misreporting is a Nash equilibrium. However, subjects' tendency for misreporting is still significant even when it does not improve their allocations [16]. Hence, the PS mechanism suffers from incentive properties. In fact other mechanisms, such as the Draft mechanism, has shown to be highly susceptible to manipulation in many real-life markets such as course allocation at business schools [9].

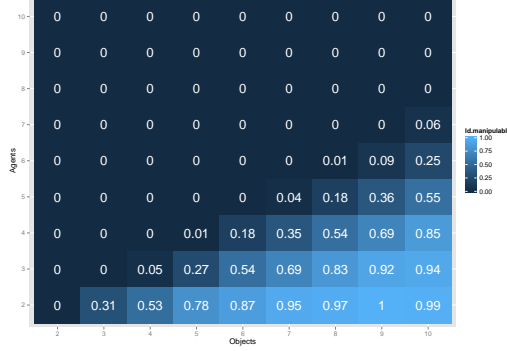


Figure 4: The fraction of *ld*-manipulable profiles under PS.

Figure 3a shows that the fraction of manipulable preference profiles goes to 1 as n or m grow. In fact, PS is almost 100% manipulable for $n > 5, m > 5$ (obviously PS is *sd*-strategyproof when $m = 2$). Another compelling observation is that for all $n < m$, the fraction of *sd*-manipulable preference profiles goes to 1 as $m - n$ grows (Figure 3b). This states that in problems where the number of objects are larger than the number of agents, an agent can strictly benefit from misreporting her preferences. Figure 4 shows that under lexicographic preferences, the fraction of *ld*-manipulable preference profiles converges to 1 even more rapidly.

7 Discussion

In this paper we conducted a comparison study of RSD and PS in order to gain further insight into their respective properties. We argued that outcomes generated by the mechanisms can be incomparable, making it difficult to determine which is the better mechanism in different settings. We then looked at some theoretical properties for the mechanisms under the assumption that agents have lexicographic preferences. Finally, we conducted an empirical comparison on the mechanisms.

Earlier work had shown that RSD is *sd*-strategyproof, ex post efficient and guarantees weak *sd*-envyfreeness. We were able to further the theoretical understanding of the mechanism by showing that it is also *ld*-envyfree. Furthermore, our empirical work showed that the actual fraction of outcomes where PS stochastically dominates RSD goes to 0 as $\frac{n}{m}$ goes to 1, raising the question as to whether the lack of *sd*-efficiency guarantees for RSD is a significant concern in practice. In contrast, PS is *sd*-efficient, *sd*-envyfree and *ld*-strategyproof for $n \geq m$. However, when $n < m$, the fraction of *sd*- and *ld*-manipulable assignments approaches 1, limiting the effectiveness of PS.

Given the theoretical and empirical results, we suggest that in many cases

	a	b	c
A_1	1/3	1/2	1/6
A_2	1/3	0	2/3
A_3	1/3	1/2	1/6

Table 6: Assignment under $PS(\succ) = RSD(\succ)$

RSD is, perhaps, a more suitable matching mechanism to consider, especially when $n \leq m$. This confirms previous studies on settings where there was an equal number of agents and objects [23, 12, 15]. We further strengthen this argument by providing an observation: At preference profiles where PS and RSD induce identical assignments, RSD is *sd*-efficient, *sd*-envyfree, and *sd*-strategyproof. However, PS may still be manipulable.

Example 6. Consider the following preference profile $\succ = ((bca), (cab), (bca))$. Table 6 shows the induced random assignment. In this case, with PS as the matching mechanism, agent 1 can misreport her preference as $\succ'_1 = (cba)$, and manipulate her assignment to $1/4(b), 1/2(c), 1/4(a)$. It is easy to see that agent 1's misreport improves her expected outcome for all utility models where $\frac{2}{6}u_1(c) > \frac{1}{4}u_1(b) + \frac{1}{12}u_1(a)$ (for example utilities 10, 9, 0 for b, c, a respectively.).

This work raises several interesting questions. From the theoretical perspective, among these open problems, providing theoretical proofs for the efficiency relations in Conjectures 1, 2, and 3 is crucial and not trivial. From the practical view, an interesting open problem is to see if there exists any randomized *ld*-efficient mechanism that satisfies strategyproofness and some notion of fairness (such as proportionality) for $n < m$. Lastly, it would be interesting to investigate if there is an extension to the PS mechanism (or its variant based on different eating speeds) that is potentially weakly *sd*-strategyproof for assigning multiple objects to each agent for $n < m$.

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